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SPECTRA OF SMASH PRODUCTS

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ABSTRACT

Let $T = R \# H$ be a smash product where H is a finite dimensional Hopf algebra. We show that ideals of T invariant under the dual H^* of H are extended from H-invariant ideals of R. This allows us to transport the study of ideals in T to invariant ideals. When the Hopf algebra is pointed the relationship between an ideal and its invariant ideal is shown to be manageable. Restricting to prime ideals, this yields results on the prime spectra of R and T. We obtain Krull relations for $R \subseteq T$ for some H, including Incomparability whenever H is commutative (or more generally when H^* is pointed after base extension). The results generalize and unify a number of results known in the context of group and restricted Lie actions.

Introduction

Let R be an associative algebra, and let H be a Hopf algebra acting on R. A smash product $R \# H$ is an algebra formed by tensoring R and H and defining an algebra structure which encodes the action of H.

The primary goal in this article is to relate the prime ideal structures of R and smash products $R \# H$. The key idea here is to look at an action of the dual Hopf algebra on the smash product as defined in [BM]. Our results effectively transport the study of ideals in a wide class of smash products to the study of ideals invariant under the action of the dual. This is exploited to obtain information about the structure of primes in $R \# H$.

The most studied examples of smash products are when H is a group algebra $k[G]$ and G acts as automorphisms of R. Perhaps most importantly, Lorenz and Passman [P2, Theorem 4.14.7] have produced a description of the primes of $R \# k[G]$ which meet R trivially, when R is a G-prime ring. [Ch2] contains a similar description for a large class of Hopf algebra crossed products when the

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(weak) action is inner on the symmetric quotient ring of R. The method of using the dual action on $R\#H$ was employed to obtain results on primes and graded primes in [CM, §6], and for divided power actions in [CQ].

Unlike common approaches to ideals in smash products, the approach taken here yields a simultaneous description of the prime spectrum of $R\#H$ in terms of the space of H-primes of R, not merely those primes of $R\#H$ having trivial intersection with the coefficient ring. Furthermore our approach is less dependent on having a "nice" basis for H and does not require dealing with X-inner actions as for smash products of group algebras and enveloping algebras. However, we must assume either that the Hopf algebra H or its dual H^* are pointed (after a finite base field extension) in order to get the most useful results on primes. Because of this we are not able to obtain proofs for all smash products of nonabelian finite groups and noncommutative restricted enveloping algebras. It is hoped that the the methods here shall yield results for these more general Hopf algebras.

We shall be concerned with how chains of primes in Spec R are related to chains in Spec T. For example, if H^* is pointed and finite dimensional or becomes so after a finite extension of the base field, (e.g. when H is commutative) we obtain Incomparability and versions of Going Up and Going Down for the extension $R \subseteq R\#H$. Two well-known examples of this type of H are when H is $k[G]$ where G is a semidirect product of a p-group by an abelian group, or when $H = u(L)$, the restricted enveloping algebra with a nilpotent augmentation ideal or where L is abelian. When H and H^* are both pointed (after base extension) we get the most comprehensive results.

For general group algebras $k[G]$ the Incomparability result is known (see [P2, Theorem 16.6] or [LP], as it is for commutative $u(L)$ [Ch1, Theorem 19]. However, the problem is as yet unsolved when L is nonabelian. When our method does yield Incomparability and other Krull relations, the results unify, generalize and simplify the known cases.

The following notation shall be used throughout. Let k denote a field and R a k-algebra. H shall denote a Hopf algebra and let U be a dense Hopf subalgebra of H^* (assuming it exists). T shall consistently denote a smash product $R\#H$.

Here is a general outline of the contents of this paper.

In §1, we review the basic machinery of Hopf actions and smash products and the action of the dual from [BM]. 1.1 summarizes some examples of finite-dimensional pointed and irreducible Hopf algebras. The crucial result in this section is Proposition 1.4, where we show that U-invariant ideals of T are extended from H-invariant ideals of R.

In §2 we specialize to prime ideals and finite-dimensional pointed Hopf algebras, establishing a correspondence between the prime spectra of R and T in Theorem 2.3. A key observation is Lemma 2.2, which amounts to the fact that H-Spec R is an orbit space of Spec R for the action of the group-like elements of H, when H is pointed. It turns out that H-primes of R correspond nicely with U-primes of T, and in fact Theorem 2.1 says that these spaces are homeomorphic for all H with dense dual U. An immediate corollary of this Theorem says that R is H-prime iff T is U-prime, extending [BM, Corollary 3.4]. When H and U are both pointed this implies that the orbit spaces of Spec $R/G(H)$ and Spec $T/G(U)$ are homeomorphic (Theorem 2.3). The correspondence is most striking if H and H^* are irreducible (Corollary 2.4), in which case Spec R is homeomorphic to Spec T. Another immediate application is given in 2.5 where we deduce a result of Lorenz and Passman on smash products of finite p-groups in characteristic p [P2, Proposition 16.4].

We then apply results and techniques of §2 to obtain Krull relations between Spec R and Spec T, assuming at least that either H or H^* becomes pointed after finite base extension. This is the goal of §3. To deal with base field extensions, and thus commutative Hopf algebras H, consider the situation where H^* becomes pointed over a finite extension of the base field in 3.1. To deal with these Hopf algebras we use the Galois theory for K/k-bialgebras and normal field extensions as explained in 3.2. The measuring K/k-bialgebra $\mathscr{H}$ of a finite normal extension is used to reduce the problem to the case of pointed H. The mechanics of this reduction are set up in 3.3 and 3.4.

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§1. Ideals in smash products and the dual action

1.1. Let us review some Hopf algebraic terminology and facts from [S] or [A] where complete details may be found. The Hopf algebra H will have structure maps $\mu, \Delta, u, \varepsilon, S$ denoting the multiplication, comultiplication, unit, counit, and antipode, respectively. $\bar{S}$ shall denote the composition inverse of S (when it exists). We shall use the following abbreviated form of Sweedler's notation for comultiplication:

$$\Delta(h) = \sum h_1 \otimes h_2, \qquad h \in H.$$

We say that a coalgebra is *irreducible* if it has a unique minimal subcoalgebra (namely $k \cdot 1$ for a Hopf algebra). A coalgebra is said to be *pointed* if all of its minimal subcoalgebras are one dimensional. Thus H irreducible implies H pointed.

Finite-dimensional Hopf algebras H are pointed if and only if each maximal ideal of H^* is of codimension 1. Similarly H is irreducible if and only if it is local. These facts follow from the correspondence between subcoalgebras of H and ideals of H^*.

Any finite-dimensional coalgebra C has a coradical filtration $C_0 \subset C_1 \subset \cdots \subset C_m = C$, where C_0 is the coradical of C. If C is pointed, then the coradical is spanned by $G(C)$, the group-like elements of C. Thus if C were a pointed Hopf algebra, C_0 is a group algebra.

Let us summarize some examples. The reader may consult [W, J] for further details concerning algebras of distributions (= hyperalgebras). Facts concerning these Hopf algebras are not needed in this paper.

EXAMPLES. It is a basic fact that group algebras are pointed and restricted enveloping algebras (and hyperalgebras in general) are irreducible.

Assume k is algebraically closed. Let H be a finite-dimensional Hopf algebra.

(i) Let $H = k[G]$. H^* is pointed iff G is the semidirect product of a group algebra of a p-group by an abelian group [P1, p. 47].

(ii) Let H be the hyperalgebra of a finite affine group scheme. H^* is pointed whenever the group scheme is finite and abelian [W, p. 70]. In fact any cocommutative Hopf algebra over an algebraically closed field is pointed [A, p. 80].

(iii) Let H be a restricted enveloping algebra of the restricted Lie algebra L. H^* is an irreducible (hence pointed) Hopf algebra provided L has a nilpotent p-map [FS, Cor. 1.3.6].

(iv) More generally H^* is irreducible if H is the hyperalgebra of a finite affine unipotent group scheme [W, p. 67].

1.2. In dealing with ideals in smash products it is natural to look at ideals generated by invariant ideals of the coefficient ring R. In some cases one might hope that ideals of $T = R \# H$ are of this form. We shall see in 1.4 exactly when this occurs.

An H-invariant ideal of R is said to be H-prime if it is prime with respect to the H-invariant ideals.

The following result is well known.

LEMMA. (i) *If I is any ideal of T, then $I \cap R$ is an H-invariant ideal of R.*

(ii) *If A is an H-invariant ideal of R, then AT is an ideal of T. Thus if P is a prime ideal of T, then $P \cap R$ is an H-prime ideal of R.*

(iii) *If H has a bijective antipode, then $AT = TA = (1 \# H)A$.*

PROOF. The first statement is elementary since $h \cdot r = \Sigma\,(1 \# h_1)r(1 \# Sh_2)$. It is easy, using the definition of crossed product multiplication, to check that $(1 \# H)A \subseteq TA \subseteq AT$. Thus AT is an ideal of T. It follows routinely from this that $P \cap R$ is H-prime.

If S is bijective, let $a \in A$, $h \in H$; then $\Sigma\,(1 \# h_2)(\bar{S}h_1 \cdot a) = a \# h$. Thus $(1 \# H)A$ contains $AT = A \# H$. □

1.3. We now give a brief exposition of some results developed by Blattner and Montgomery [BM] to prove duality for certain Hopf algebra smash products. While we do not use their duality theorem *per se*, it led us to suspect that an examination of dual actions on smash products could be fruitful.

Let $T = R \# H$ be a smash product. In this section U shall denote a dense Hopf subalgebra of the dual H^* of H. The tacit assumption on H is of course that such a U exists. It is known that such a U exists if H is "residually finite" ("proper"), i.e., the intersection of the cofinite ideals of H is zero (cf. [A]). Examples of such H are the group algebra of a finitely generated nilpotent group and the enveloping algebra of a solvable Lie algebra [cf. BM, §4]. We are mainly interested in finite-dimensional H, in which case $H^* = U$.

We use a map defined in [BM, p. 156] to obtain an action of U on T as follows. Let $m, h \in H$ and $u \in U$. The map $\lambda : U \to \mathrm{End}_k(T)$ is an algebra map making T a U-module and is given by

$$\lambda(u)(r \# h) = r \# u \rightharpoonup h,$$

where $\Sigma\, h_1 \otimes h_2$ shall denote the comultiplication $\Delta(h)$ and the action of U on H is given by $u \rightharpoonup h = \Sigma\,\langle u, h_2\rangle h_1$.

This action extends to the algebra map $\Lambda : H \# U \to \mathrm{End}_k(T)$, which makes T a left $H \# U$-module. Λ is defined by

$$\Lambda(m \# u)(r \# h) = (1 \# m)(r \# u \rightharpoonup h),$$

which we shall write as $(m \# u) \rightharpoonup (r \# h)$. The reader may observe that this is the restriction of the map α [BM, p. 159] to $(1 \# H) \# U = H \# U$. Furthermore note that the elements of $H \# U$ act as right R-module homomorphisms of T:

$$\begin{aligned}\Lambda((m \# u)(r \# h)s) &= \sum (1 \# m)(r(h_1 \cdot s) \# u \rightharpoonup h_2)\\ &= \sum (1 \# m)(r((u \rightharpoonup h)_1 \cdot s) \# (u \rightharpoonup h)_2 = (\Lambda(m \# u)(r \# h))s,\end{aligned}$$

where the second equality follows from the identity $\Delta(u \rightharpoonup h) = \Sigma\, h_1 \otimes u \rightharpoonup h_2$.

The action of U on H given by $u \rightharpoonup h$ as above also extends to make H an $H \# U$-module via the algebra map $L : H \# U \to \mathrm{End}_k(H)$ given by

$$L(m \# u)(h) = m(u \rightharpoonup h),$$

which we shall denote by $(m \# u) \rightharpoonup h$.

The following lemma will allow us to use the action of $H \# U$ on T to "pick out" coefficients of elements of T:

LEMMA [BM]. *Suppose the Hopf algebra H has a bijective antipode and suppose U is a dense Hopf subalgebra of H^*. Then $L(H \# U)$ is a dense ring of linear transformations of H.* □

1.4. The Proposition below is the key result relating ideals of R to the smash product.

LEMMA. *If A is an H-invariant ideal of R, then AT is a U-invariant ideal of T. Therefore AT is an $H \# U$-submodule of T.*

PROOF. It is evident from the definition of the action via Λ of $H \# U$ on T that $AT = A \# H$ is invariant under the image in End T of $U = 1 \# U$. And here $H = H \# 1$ acts on T by left multiplication. Thus $A \# H$ is $H \# U$-invariant. □

DEFINITION. Let H act on R. For ideals A of R, let $(A : H)$ denote the largest H-invariant ideal contained in A.

It is easy to check that $(A : H) = \{r \in R \mid h \cdot r \in A \text{ for all } h \in H\}$.

PROPOSITION. *Suppose that H has bijective antipode. Let I be an ideal of $R \# H$. Then $(I : U) = (I \cap R)T$.*

PROOF. Let I be an ideal of T and let $\alpha \in (I : U)$. Fix a basis $\{h_i\}$ for H and write $\alpha = \Sigma\,(1 \# h_i)(a_i \# 1)$ with $\alpha \in R$ (Lemma 1.2(iii)). By the previous Lemma there exists $f_i \in H \# U$ with $f_i \rightharpoonup h_j = \delta_{ij}$. Thus $f_i \rightharpoonup \alpha = a_i \in (I : U) \cap R$, using the fact that $H \# U$ acts as right R-module homomorphisms (1.3).

Therefore $(I \cap R)T$ contains $(I : U)$. The reverse inclusion follows from the previous Lemma. □

1.5. A standard Zorn's Lemma argument establishes the next lemma, which shall be used freely in the remainder of the paper.

LEMMA. *Let H be a finite dimensional Hopf algebra and let Q be an H-prime ideal of R. Then there exists a prime ideal Q' of R with $(Q' : H) = Q$.* □

§2. Invariant ideals and spectra

2.1. We consider the Zariski topology as usual on Spec R. We analogously define a topology on H-spec R, the set of H-prime ideals of R, where a basic closed set is one of the form $\{P \in H\text{-Spec } R \mid B \subseteq P\}$ where B is an subset of R. Obviously B may be taken to be an H-invariant ideal of R.

THEOREM. *Suppose H has bijective antipode and dense dual $U \subseteq H^*$. U-Spec T is homeomorphic to H-Spec R where the map is given by sending $P \in U$-Spec T to $P \cap R$.*

PROOF. Suppose that P is U-prime. Then $P = (P : U) = (P \cap R)T$ and $P \cap R$ is an H-prime ideal of R by Lemma 1.2. Thus the map is well-defined and injective.

Given $A \in H$-Spec R, let P' be an ideal of T maximal such that $P' \cap R = A$, and then let $P = (P : U)$. It follows routinely that P' is a prime ideal of T. Therefore $P \in U$-Spec T.

We observe that $(P' : U) = P = (P : U) = (P' \cap R)T = AT$ and hence $P \cap R = A$. Plainly, the inverse map can be defined by sending A to AT. Thus the map is bijective.

It is clear that the map sending P to $P \cap R$, and its inverse are inclusion-preserving. It is pedestrian to verify that the map is a homeomorphism since we only need to consider invariant ideals in defining the topologies on U-Spec T and H-Spec R. □

We digress for a moment to point out a strengthening of [BM, Corollary 3.4] where it is shown that if R is prime then T is U-prime.

COROLLARY. *Suppose H has a bijective antipode and dense dual U. Then T is U-prime if and only if R is H-prime.*

PROOF. If T is U-prime then $0 = (P : U) = (P \cap R)T$ for some $P \in \operatorname{Spec} T$ (take P to be maximal such that $P \cap R = 0$). Thus $P \cap R = 0$ is H-prime by Lemma 1.2. The converse is similar using the action of U on T. □

2.2. Parts (i) and (ii) of the next Lemma extend [Ch1, Lemma 1], [Ch2, Lemma 1.8] and well-known results about group actions.

LEMMA. *Let H be a finite-dimensional pointed Hopf algebra acting on the algebra R.*

(i) *For any ideal P of R, $(\bigcap_{x \in G(H)} x \cdot P)^m \subseteq (P : H)$ for some m.*

(ii) *Suppose $P_1 \subseteq P_2 \in \operatorname{Spec} R$. Then $(P_1 : H) = (P_2 : H)$ iff $P_1 = P_2$.*

(iii) *If $Q \in H$-Spec R, then there exists $P \in$ Spec R with $(P : H) = Q$, and the set of primes minimal over Q is precisely the orbit of P under the action of $G(H)$.*

(iv) *If $Q_1 \subseteq Q_2 \subseteq Q_3 \in H$-Spec R and $P_2 \in$ Spec R with $(P_2 : H) = Q_2$, then there exist P_1 and P_3 with $P_1 \subseteq P_2 \subseteq P_3 \in$ Spec R with $(P_i : H) = Q_i$.*

PROOF. To prove (i) we may suppose $(P : H) = 0$. Let J denote $\bigcup_{x \in G(H)} (x \cdot P)$, the largest $G(H)$-invariant ideal contained in P. Note that each $x \cdot P$ is a prime ideal of R if P is prime.

Let $H_0 \subset H_1 \subset \cdots \subset H_m = H$ denote the coradical filtration of H. We show by induction on j that $H_j \cdot J^n \subset J$ if $n > j$. If $j = 0$, $H_j = kG(H)$ so the result is clear from the definition of J. Assume $j > 0$ and note that

$$\begin{aligned} H_j \cdot (J^n) &= H_j \cdot (JJ^{n-1}) \\ &\subseteq (H_0 \cdot J)(H_j \cdot J^{n-1}) + (H_j \cdot J)(H_0 \cdot J^{n-1}) + (H_{j-1} \cdot J)(H_{j-1} \cdot J^{n-1}) \\ &\subseteq J, \end{aligned}$$

where the first inclusion holds because H measures R. The last inclusion holds because $H_0 = kG(H)$ and by induction. We have shown that $H_m \cdot J^{m+1} = H \cdot J^{m+1} \subseteq J$; thus $J^{m+1} \subseteq (J : H) \subseteq (P : H) = 0$. We have proved (i).

(ii) Suppose that $(P_2 : H) \subseteq (P_1 : H)$. By (i) $(\bigcap_{x \in G(H)} x \cdot P_2)^n \subseteq (P_1 : H) \subseteq P_1$, whence $x \cdot P_2 \subseteq P_1 \subseteq P_2$ for some $x \in G(H)$. Now, repeatedly applying the automorphism x to the inclusion $x \cdot P_2 \subseteq P_2$ we obtain

$$P_2 = x^m \cdot P_2 \subseteq \cdots \subseteq x^2 \cdot P_2 \subseteq x \cdot P_2 \subseteq P_1,$$

where m is the order of x. The reverse implication is obvious.

The first statement of (iii) is just Lemma 1.5. Here we may assume $Q = 0$. If $I \supseteq Q$ is any prime, $I \supseteq x \cdot P$ for some x. Therefore the minimal primes are among the $x \cdot P$. To see that $x \cdot P$ is minimal, choose a minimal prime, say $y \cdot P$ ($y \in G(H)$), contained in $x \cdot P$. That $y \cdot P = x \cdot P$ now follows from the fact that $(x \cdot P : H) = (y \cdot P : H) = (P : H)$ and part (ii).

(iv) Choose P_i ($i = 1, 3$) as in (iii), with $(P_i : H) = Q_i$. Since P_2 is prime, (i) implies that some conjugate $x \cdot P_2$ is contained in P_3. Now $P_2 \subseteq x^{-1} \cdot P_3$, so by relabeling we have $P_2 \subseteq P_3$ with $(P_3 : H) = Q_3$. Similarly, we choose a conjugate of P_1 and relabel to obtain a prime $P_1 \subseteq P_2$ with $(P_1 : H) = Q_1$. □

2.3. THEOREM. *If H is finite dimensional and H and U are both pointed, then* (Spec T)/$G(U)$ *is homeomorphic to* (Spec R)/$G(H)$.

PROOF. It is apparent from the previous Lemma that the fibers of the map

Spec $T \to U$-Spec T sending P to $(P : U)$ consist of orbits of minimal primes over U-primes. Thus the conclusion follows immediately from Theorem 2.1. □

2.4. The following corollary applies when H is $u(L)$, L finite dimensional with nilpotent p-map, or, more generally, a hyperalgebra of a finite unipotent group scheme.

COROLLARY. *If H and its dual H^* are both irreducible, then* Spec T *is homeomorphic to* Spec R.

2.5. We state an application. This result is due to Lorenz and Passman [P2, 4.16.4], which now has a relatively simple proof.

COROLLARY [LP]. *If $R \# k[G]$ is a smash product of the finite p-group G over the G-prime ring R and k is a field of characteristic $p > 0$, then $R \# k[G]$ has a unique minimal prime ideal which is necessarily nilpotent.*

PROOF. Let $H = k[G]$. If R is H-prime then T is U-prime. Thus since $G(U)$ is trivial (since H is local) T has a unique minimal prime by Lemma 2.2(ii). The nilpotence is a result of Lemma 2.2(i). □

REMARK. The Lorenz–Passman result actually deals with crossed products $R * G = R \#_t k[G]$ with invertible cocycle t (i.e. t has values in the units of R). The following argument shows that H^*-invariant ideals are extended from R.

Let $H = kG$ and write $H^* = \Sigma\, kp_x$ where the p_x are dual to the group elements $x \in G$. Then H^* acts on $R * G$ with $p_y \cdot (r \# x) = (r \# y\delta_{x,y})$. If $\Sigma\, r_x \# x$ is a typical element of the H^*-invariant ideal $I \subseteq R * G$, we see that $r_y \# y \in I$. And since I is an ideal, we have

$$(r_y \# y)(1 \# y^{-1}) = t(y, y^{-1})r_y \in I \cap R \qquad \text{for all } y \in G.$$

Thus, since t has values in the units of R, we conclude that $I = (I \cap R)T$.

With this fact in hand arguments extend to the case where an invertible cocycle is present. In particular the proof of 2.5 goes through when an invertible cocycle t is present and $H = k[G]$.

§3. Prime ideal relations

In this section we enhance our techniques to deal with Hopf algebras which become pointed under finite base field extension. H is said to be *virtually* *pointed* if $H \otimes_k K$ is pointed as a K-coalgebra for some finite extension $K \supseteq k$. Here we shall say that H is *split* by K. We shall, without loss, assume that K/k is a finite normal extension.

We shall habitually write VK for $V \otimes_k K$ whenever V is a vector space over k, and K is a field extension of k.

3.1. The main examples of virtually pointed Hopf algebras are finite-dimensional cocommutative ones. The following finite *NullstellenSatz* is well-known.

LEMMA. *Let H be a finite-dimensional commutative algebra. Then H^* is virtually pointed.*

PROOF. Let K be the (finite normal) subfield of the algebraic closure $\hat{k}$ of k generated by the residue fields of the algebra H.

Note that $\mathrm{Hom}_{k\text{-alg}}(H, \hat{k})$ corresponds naturally with $\mathrm{Hom}_{K\text{-alg}}(HK, \hat{k})$ (given $\phi : H \to \hat{k}$, define the K-linear map $\phi_K : HK \to \hat{k}$ by $\phi_K(h \otimes a) = \phi(h)a$). Also note that every maximal ideal of HK is the kernel of some K-algebra map $\phi_K : HK \to \hat{k}$. The corresponding map ϕ has image (a subfield of $\hat{k}$ since H is finite dimensional) in K by construction. Now the correspondence immediately implies ϕ_K also has image K. It follows that any residue field of HK is isomorphic to K, as desired. □

3.2. In preparation for the main result concerning virtually pointed Hopf algebras, let us summarize some field theory from [A, chapter 5]. This notation shall persist for the remainder of the paper.

Suppose K is a finite normal field extension of k. $\mathscr{H} = \mathrm{End}_k(K)$ may be considered a k-algebra and a K-vector space where K acts by multiplication in K, and the product is defined to be *composition* of functions. $\mathscr{H}$ is a K-coalgebra measuring K with comultiplication determined by the equation

$$f(ab) = \sum (f_1 \cdot a)(f_2 \cdot b)$$

where $\Delta f = \Sigma f_1 \otimes f_2$, and counit determined by $\varepsilon(f) = f(1)$, and $f \in \mathscr{H}$, $a, b \in K$.

$\mathscr{H}$ is a cocommutative pointed K-coalgebra and $G(\mathscr{H})$ is the Galois group $\mathrm{Gal}(K/k)$. But $\mathscr{H}$ is not quite a Hopf algebra because the counit is not necessarily a k-algebra morphism nor is K central in $\mathscr{H}$.

$\mathscr{H}$ is an example of a *K/k bialgebra*. It plays a role tantamount to the Galois group for normal field extensions. In fact $G(\mathscr{H})$ corresponds to the separable part of the extension. On the other hand, being pointed and cocommutative,

$\mathcal{H}$ is the direct sum as a coalgebra of irreducible coalgebras, one for each element of $G(\mathcal{H})$. The irreducible component containing the identity corresponds to the purely inseparable part of the extension.

We say that $\mathcal{H}$ *acts* on a K-algebra A if A is an $\mathcal{H}$-module and is measured by the coalgebra $\mathcal{H}$. This will occur below when $A = RK = R \otimes_k K$ and $\mathcal{H}$ acts on the second factor. Observe that invariant ideals can be defined by

$$(I : \mathcal{H}) = \{a \in I \mid f(a) \in I \text{ for all } f \in \mathcal{H}\}$$

for ideals I of A, just as for Hopf algebra actions.

3.3. Here we are concerned with relations between prime ideals in the extension $R \subseteq RK$. Some of the results in the Lemma below are subsumed by results in [RS]. However, the situation here is much simpler than for general centralizing extensions, and our approach is parallel to the one used for pointed Hopf algebras actions. Also we need to deal with H-primes as in (v) below. Therefore we have decided to start more or less from scratch and use the K/k bialgebra action as in 3.2 to deal with the base extension problem.

LEMMA. *Assume the notation of* 3.2. *Consider the action of $\mathcal{H}$ on RK and let I denote an ideal of RK.*

(i) *If* $I \in \operatorname{Spec} RK$ *then* $I \cap R \in \operatorname{Spec} R$.

(ii) $(I : \mathcal{H}) = (I \cap R)K$.

(iii) $(\bigcap_{g \in G(\mathcal{H})} g \cdot I)^m \subseteq (I : \mathcal{H})$ *for some* m.

(iv) *Let* $I_1 \subseteq I_2 \in \operatorname{Spec} RK$. *Then* $(I_1 : \mathcal{H}) = (I_2 : \mathcal{H})$ *iff* $I_1 = I_2$.

(v) *Let* $I_1 \subseteq I_2 \in HK\text{-}\operatorname{Spec} RK$. *Then* $(I_1 : \mathcal{H}) = (I_2 : \mathcal{H})$ *iff* $I_1 = I_2$.

PROOF. (i) is elementary.

(ii) Let $\Sigma\, r_i \otimes b_i (I : \mathcal{H})$ where the b_i are k-linearly independent. Since $\mathcal{H} = \operatorname{End}_k K$, there is some $f \in \mathcal{H}$ with $f(b_i) = 1$ if $i = j$ and zero otherwise. Thus $f(\Sigma\, r_i \otimes b_i) = r_j \otimes 1 \in I$, for any fixed j. Hence $r_i \in I \cap R$ and thus $(I : \mathcal{H})$ is contained in $(I \cap R)K$. The reverse inclusion is immediate so we are finished with (ii).

(iii) Since $\mathcal{H}$ is a finite-dimensional coalgebra over K, it has a coradical filtration

$$KG(\mathcal{H}) = \mathcal{H}_0 \subset \mathcal{H}_1 \cdots \subset \mathcal{H}_m = \mathcal{H}$$

for some m. Noting that $G(\mathcal{H})$ is a multiplicative subgroup of $\mathcal{H}$, we can imitate the proof of Lemma 2.2(i), whose proof goes through with K in place of k. This yields $(\Pi_{g \in G(\mathcal{H})}\, g \cdot I)^m \subseteq (I : \mathcal{H})$ for some m, where the product ranges over $g \in G(\mathcal{H})$.

(iv) This is essentially the same as Lemma 2.2(iv). Suppose that $(I_2 : \mathcal{H}) \subseteq$

$(I_1 : \mathscr{H})$. By (ii) $(\bigcap_{\theta \in G(\mathscr{H})} \theta \cdot I_2)^n \subseteq (I_1 : \mathscr{H}) \subseteq I_1$, whence $\theta \cdot I_2 \subseteq I_1 \subseteq I_2$ for some $\theta \in G(\mathscr{H})$. Now

$$I_2 = \theta^m \cdot I_2 \subseteq \cdots \subseteq \theta^2 \cdot I_2 \subseteq \theta \cdot I_2 \subseteq I_1,$$

where m is the order of the automorphism θ. This reverse implication is obvious.

(v) The proof in (iv) goes through once we show that $\theta \cdot I$ is HK-invariant if I is also. To do this let $h \otimes c \in HK$ and $r \otimes d \in I$, and observe that

$$\begin{aligned}(h \otimes c) \cdot (\theta \cdot (r \otimes d)) &= (h \otimes c) \cdot (r \otimes \theta(d)) \\ &= (h \cdot r \otimes c\theta(d)) \\ &= (h \cdot r \otimes \theta(\theta^{-1}(c)d)) \\ &= \theta \cdot (h \otimes \theta^{-1}(c)) \cdot (r \otimes d) \\ &\in \theta \cdot (HK \cdot I) \\ &\in \theta \cdot I\end{aligned}$$

Thus I is HK-invariant. □

3.4. Concerning base extension and primes we further have the following

LEMMA. *Consider the extension* $R \subseteq RK$, *with notation as in* 3.2.

(i) *If* $P \in \operatorname{Spec} R$, *then there exists* $I \in \operatorname{Spec} RK$ *with* $(I : \mathscr{H}) = PK$, *and* $I \cap R = P$.

(ii) *Suppose* $P_1 \subseteq P_2 \subseteq P_3$ *are in* Spec R *and* $I_2 \in \operatorname{Spec} RK$ *with* $I_2 \cap R = P_2$. *Then there exists a chain* $I_1 \subseteq I_2 \subseteq I_3 \in \operatorname{Spec} RK$ *with* $(I_i : \mathscr{H}) = P_i K$, *and* $I_i \cap R = P_i$.

PROOF. (i) Take I to be maximal (Zorn) such that $(I : \mathscr{H}) = PK$. Then I is prime. That $I \cap R = P$ follows from (ii) in the previous Lemma.

(ii) Let $J_1 \in \operatorname{Spec} RK$ with $J_1 \cap R = P_1$ using (i). Then $\theta \cdot J_1 \subseteq I_2$ for some $\theta \in G(\mathscr{H})$ using (iii) in the Lemma and the fact that I_2 is prime. Setting $I_1 = \theta \cdot J_1$ it is easy to check that $(I_1 : \mathscr{H}) = (J_1 : \mathscr{H}) = P_1 K$, and thus $I_1 \cap R = P_1$. Similarly choose $J_3 \in \operatorname{Spec} RK$ with $J_3 \cap R = P_3$. Then $\theta \cdot I_2 \subseteq J_3$ for some $\theta \in G(\mathscr{H})$. Now set $I_3 = \theta^{-1} \cdot J_3$.

3.5. We say that $P \in \operatorname{Spec} T$ *lies over* $Q \in \operatorname{Spec} R$ if $(Q : H) = P \cap R$. Note that if H is pointed, Q is a prime minimal over $P \cap R$ by Lemma 2.2(iii). The following is our basic lying over/under result.

PROPOSITION. *Let $T = R \# H$ where H is a finite-dimensional Hopf algebra with dual $U = H^*$.*

(i) *If $P \in \operatorname{Spec} T$ there exists $Q \in \operatorname{Spec} R$ with P lying over Q.*

(ii) *If $Q \in \operatorname{Spec} R$ there exists $P \in \operatorname{Spec} T$ with P lying over Q. When U is pointed, any such P is a minimal prime over $(Q : H)T$.*

PROOF. (i) Q exists by Lemma 1.5 since $P \cap R$ is an H-prime ideal of R.

(ii) By 1.5 and 2.1, we see that P exists as in the statement, with $(P : U) = (P \cap R)T = (Q : H)T$. Using Lemma 2.2(iii) we obtain the minimality claim. □

3.6. Here is the main result.

THEOREM. *Let $T = R \# H$ be a smash product with H finite dimensional. Let U denote the dual H^* of H.*

(i) (Incomparability) *Suppose that U is virtually pointed. If $P_1 \subsetneqq P_2$ are prime ideals of T, then $P_1 \cap R \neq P_2 \cap R$.*

The following Going Up/Down statements hold:

(ii) *Let $P_1 \subseteq P_2 \subseteq P_3 \in \operatorname{Spec} T$ with P_2 lying over $Q_2 \in \operatorname{Spec} R$. If H is virtually pointed, then there exist $Q_i \in \operatorname{Spec} R$ with P_i lying over Q_i $(i = 1, 3)$ and $Q_1 \subseteq Q_2 \subseteq Q_3$.*

(iii) *Let $Q_1 \subseteq Q_2 \subseteq Q_3 \in \operatorname{Spec} R$ with $P_2 \in \operatorname{Spec} T$ lying over Q_2. If U is virtually pointed then there exist $P_i \in \operatorname{Spec} T\, (i = 1, 3)$ with P_i lying over Q_i and $P_1 \subseteq P_2 \subseteq P_3$.*

PROOF. (i) Suppose first that U is pointed and let $P = P_i$ $(i = 1, 2)$ be primes as in the statement. By Proposition 1.4 $(P : U) = AT \in U\text{-}\operatorname{Spec} R$, where $A = P \cap R$.

Suppose that $P_1 \cap R = P_2 \cap R$ so that $A_1T = (P_1 : U) = A_2T = (P_2 : U)$. This forces $P_1 = P_2$ by Lemma 2.2(ii).

Now we deal with the case where UK is pointed. Suppose that $P_1 \cap R = P_2 \cap R$. Let $I_i \in \operatorname{Spec} TK\, (i = 1, 2)$ be as asserted in Lemma 3.4, with $I_i \cap T = P_i$ and $I_1 \subseteq I_2$.

Observe that

$$(I_i \cap RK : \mathscr{H}) = (I_i \cap R)K = (P_i \cap R)K.$$

Thus our supposition yields $(I_1 \cap RK : \mathscr{H}) = (I_2 \cap RK : \mathscr{H})$, and this ideal is HK-prime since I_i is prime. By Lemma 3.3(v) we have $I_1 \cap RK = I_2 \cap RK$. By the pointed case this implies $I_1 = I_2$. Finally, intersecting with T yields $P_1 = P_2$.

(ii) When H is pointed the conclusion is immediate from Lemma 2.2(iv), since the $P_i \cap R$ are H-prime.

Assume that HK is pointed and let $G = G(HK)$. Then by Lemma 3.4, there exist primes $I_1 \subseteq I_2 \subseteq I_3 \in \operatorname{Spec} TK$ with $I_i \cap T = P_i$. By Lemma 3.5(i) there exists $J = J_2 \in \operatorname{Spec} RK$ with $I = I_2$ lying over J. Note also that J and its G-conjugates are the primes minimal over $I \cap RK$.

We may choose J so that $J \cap R = Q = Q_2$ by the following argument: First observe that

$$(QK : HK) = (Q : H)K = (P \cap R)K \subseteq (I \cap RK) \subseteq J.$$

Hence by Lemma 2.2 and the fact that J is prime, we get $x \cdot QK \subseteq J$ for some $x \in G$. Thus $QK \subseteq x^{-1} \cdot J$, and evidently $(x^{-1} \cdot J : HK) = (J : HK)$. Replace J with $x^{-1} \cdot J$ so now $QK \subseteq J$. Next choose $Q' \in \operatorname{Spec} RK$ such that $(Q' : \mathscr{H}) = QK$ using Lemma 3.4(i), and note that Q' may be chosen so that $Q' \subseteq J$ by Lemma 3.3(ii)(iii). Here we have $J = Q'$ since J is a minimal prime over $I \cap RK$. Hence $Q = Q' \cap R = J \cap R$.

By the pointed case there exists a chain $J_1 \subseteq J_2 \subseteq J_3$ in Spec RK with I_i lying over J_i. Set $Q_i = J_i \cap R$ $(i = 1, 3)$. Let us drop the subscript i. It remains to show that $P \cap R = Q$. By the choice of J, we have $(J : HK) = I \cap RK$. It is now easily verified that

$$(J \cap R : H) = (J : HK) \cap R = I \cap RK \cap R = P \cap R;$$

and since $J \cap R = Q$, we obtain $P \cap R = (Q : H)$ as required.

(iii) This is similar to (ii):

Assume first that U is pointed. Since $A_i = (Q_i : H)T$ is a U-prime, there are $P_i \in \operatorname{Spec} T$ $(i = 1, 3)$ minimal over A_i with $P_1 \subseteq P_2 \subseteq P_3$ using Lemma 2.2(iv). It is apparent that P_i lies over Q_i (Lemma 3.4(ii)).

Moving to the case where UK is pointed, we choose $J_i \in \operatorname{Spec} RK$ with $J_1 \subseteq J_2 \subseteq J_3$ with $J_i \cap R = Q_i$.

Dropping the subscript 2, let $P = P_2$, etc., and choose $I \in \operatorname{Spec} TK$ with I lying over J.

Without loss we may assume that $I \cap T = P$ by the following argument: First observe that

$$(PK : UK) = (PK \cap RK)TK = (P \cap R)TK \subseteq (I \cap R)TK \subseteq I.$$

Hence by replacing I with a $G(UK)$-conjugate if necessary (as in (ii)), we have $PK \subseteq I$ and thus $P \subseteq I \cap T$. Observe that

$$(I \cap T) \cap R = I \cap TK \cap R = (J : HK) \cap R = (J \cap R : H) = (Q : H) = P \cap R,$$

where the last equality is the hypothesis that P lies over Q. As $I \cap T$ and P are prime, we conclude that $P = I \cap T$, using (i).

By the pointed case there exist $I_1 \subseteq I_2 \subseteq I_3 \in \operatorname{Spec} TK$ with I_i lying over J_i. Set $P_i = I_i \cap T$ $(i = 1, 3)$. Again we drop the subscript i. It remains to show that $P \cap R = (Q : H)$. By the choice of I, we have $(J : HK) = I \cap RK$. It now follows that

$$\begin{aligned} P \cap R &= I \cap T \cap R = I \cap RK \cap R \\ &= (J : HK) \cap R = (J \cap R : H) \\ &= (Q : H). \qquad \square \end{aligned}$$

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